Inverse Function Theorem Exercises

Question 1 Consider the following transformations $f : \mathbb{R}^2 \to \mathbb{R}^2$:

i) $\begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}$ ii) $\begin{cases} u = x^2 \\ v = y/x \end{cases}$ iii) $\begin{cases} u = x^2 + 2xy + y^2 \\ v = 2x + 2y \end{cases}$ iv) $\begin{cases} u = x + y \\ v = 2xy^2 \end{cases}$

For each transformation above:

a) Compute $\det(\mathbf{D}f)$.

b) Find (if possible), using the inverse function theorem, regions where the transformation is locally invertible. If no such regions exist, explain why.

c) Find the image of the square $D = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$ under the transformation. Is the image of D locally diffeomorphic to D?

Question 2 This is problem 8 section 3.5 in the textbook. Is the transformation

$$\begin{cases} u = x + xyz \\ v = y + xy \\ w = z + 2x + 3z^2 \end{cases}$$

invertible near the point $(x, y, z) = (0, 0, 0)$?

Question 3 This is problem 9 section 3.5 in the textbook. Is the transformation

$$\begin{cases} u = \dfrac{x^2 - y^2}{x^2 + y^2} \\ v = \dfrac{xy}{x^2 + y^2} \end{cases}$$

invertible near the point $(x, y) = (0, 1)$?